ALGEBRA QUALIFYING EXAM, JANUARY 2020

A) Attempt to solve 5 of these problems. (You can omit one problem.)
B) Indicate clearly which problem you are omitting.

1. (a) Let G be a finite group such that for every positive integer n, G has at most one subgroup of order n. Show that G is cyclic. (Hint: You might first prove this when G is a p-group.)
   (b) Find a group G of some order n and a positive integer d dividing n such that G has no subgroup of order d. (Justify your answer.)

2. Let G be a group of order p^2q, where p and q are primes with $p < q$. Prove that either G has a normal q-Sylow subgroup or G is isomorphic to the alternating group A_4.

3. (a) Let A and B be finite abelian groups. Suppose that for every positive integer n, the groups A and B have the same number of elements of order n. Prove that A and B are isomorphic.
   (b) Let A and B be finitely generated abelian groups. Suppose that A is isomorphic to a subgroup of B, and B is isomorphic to a subgroup of A. Prove that A and B are isomorphic.

4. (a) Let k be a field and let $R = k + x^2k[x]$ be the subring of $k[x]$ consisting of polynomials $f = \sum a_i x^i$ with $a_1 = 0$ (no linear term). Show that every nonzero nonunit of R has a factorization into irreducible elements. Prove or disprove that R is a unique factorization domain.
   (b) Suppose that R is a Noetherian integral domain and every finitely generated torsion-free R-module is free. Show that R is a principal ideal domain.

5. Let R be a commutative Noetherian ring.
   (a) Prove that if J is any non-prime ideal of R, then there exist $a, b \notin J$ such that $(J + Ra)(J + Rb) \subset J$.
   (b) Using (a) and the Noetherian property, prove that for any ideal I of R, there exist prime ideals $P_1, \ldots, P_m$ of R such that
   $$P_1 P_2 \cdots P_m \subset I.$$
   (c) Prove that R has only finitely many minimal prime ideals (minimal with respect to set inclusion). (Hint: Look at the zero ideal).

6. Let R be a ring and $0 \to M' \to M \to M''$ an exact sequence of left R-modules. Suppose N is another left R-module.
   (a) Show that there is an exact sequence of abelian groups
   $$0 \to \mathrm{Hom}_R(N, M') \to \mathrm{Hom}_R(N, M) \to \mathrm{Hom}_R(N, M'').$$
   (b) Give an example for which $M \to M''$ is surjective but the corresponding homomorphism $\mathrm{Hom}_R(N, M) \to \mathrm{Hom}_R(N, M'')$ is not surjective.